Vector, scalar and tensor

Magnitude – absolute value of a physical quantity is known as its magnitude.

Ex – (i) Let work, $W = 24\,J$

magnitude of work $= 24\,J$

(ii) Let force, $\vec{F} = -15\,N$ in x-axis

magnitude of force $= 15\,N$

(iii) Let refractive index, $\mu = 1.5$

magnitude of refractive index $= 1.5$

⇒ Usually magnitude includes numerical value and unit, but not always.

Specific direction – a direction which can be specify in space is known as specific direction.

On the basis of magnitude and direction physical quantities are classified as three types

(a) vector

(b) scalar

and (c) tensor.

Scalar – a physical quantity which complete representation requires only magnitude not a specific direction is called scalar.

Ex – distance, speed, mass, charge, work, energy, power, electric potential, electric flux, magnetic flux etc.

Representation of scalar

(i) Work $W = 24\ J$

(ii) Speed $v = 2\ m s^{-1}$

(iii) Electric potential
$V = 20$ volt

(iv) Electric flux
$\phi_e = 120\ V m^{-1}$

(v) Magnetic flux
$\phi_m = 2.0$ weber

⇒ A scalar may have a direction, but the direction is not specific.

Ex - Electric current and time have direction but the direction is not ~~specfic~~ specific, so they are not vectors.

⇒ Two scalars of same dimensional formula are added/subtracted by simple algebric law.

Ex - Let kinetic energy $K = 20 J$
potential energy $U = 16 J$

then total mechanical energy

$$E = K + U$$
$$= 20J + 16J = 36J$$

⇒ Two scalars are multiplied and divided by simple algebric law.

Ex - Let mass $m = 10$ kg, Work $W = 24 J$
time $t = 4 s$

$$\text{Power} = \frac{\text{Work}}{\text{time}} = \frac{24 J}{4 s} = 6 W.$$

Vector – a physical quantity which representation requires magnitude as well as a specific direction is called vector.

Ex – displacement, velocity, acceleration, force, momentum, impulse, torque, electric field, magnetic field, angular velocity etc.

Notation of a vector – Vector A is denoted as $\vec{A}$ or $\boldsymbol{A}$ (in bold italic letter).

Magnitude of a vector – absolute value of a vector is known as its magnitude. Magnitude is never negative, so magnitude of a vector is represented by taking modulus of a vector.

Ex – $|\vec{A}|$ = magnitude of vector $\vec{A}$

$|\vec{R}|$ = magnitude of vector $\vec{R}$

Let Velocity $\vec{V} = 10\ m\,s^{-1}$ in east

then $|\vec{V}| = 10\ m\,s^{-1}$

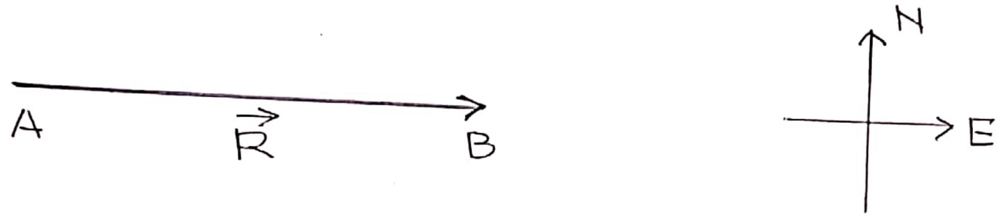
Force $\vec{F} = -15\ N$ in west

then $|\vec{F}| = 15\ N$

⇒ Negative sign comes to clear direction, it is not a part of magnitude.

Geometrical representation of a vector

A vector is geometrically represented by an arrow headed line segment. Length of the line segment gives magnitude where the direction of arrow gives direction of a vector.



A = initial point/tail

B = terminal point/head

$|\vec{R}| = AB$

Direction of $\vec{R}$ is east.

Line of action of a vector –

A straight line along the line segment of a vector is known as line of action of the vector

$X'X$ = line of action of force.

Conditions in which a vector doesn't change

(i) On translating parallely

(ii) On rotating with an angle $\theta = 2n\pi$, $n = 1, 2, 3, \ldots$

Conditions in which a vector changes

(i) On translating non parallely

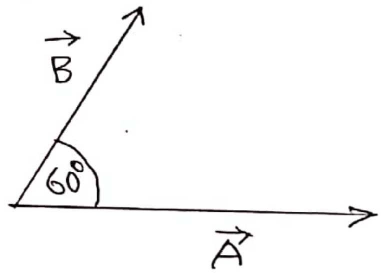
(ii) On rotating with an angle $\theta \neq 2n\pi$, $n = 1, 2, 3, \ldots$

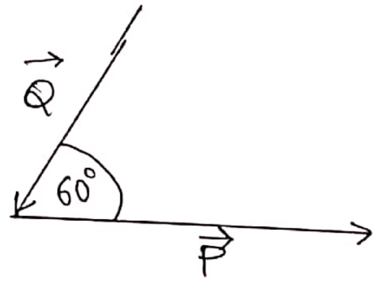
Angle between two vectors

When either initial point or terminal point of two vectors coincide, then smaller angle between the linesegments of vectors is known as angle between two vectors.

Angle between $\vec{A}$ and $\vec{B}$

$\theta = 60^\circ$

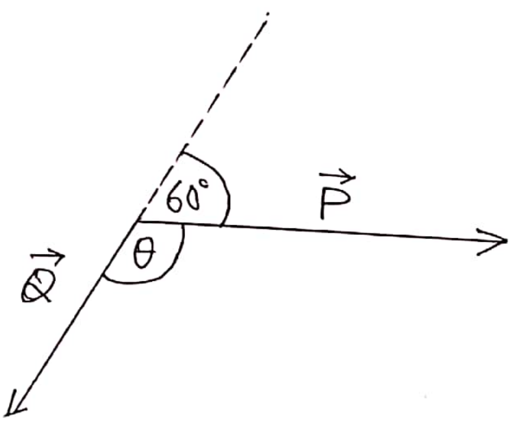




Angle between $\vec{P}$ and $\vec{Q}$

$\theta = (180^\circ - 60^\circ)$

$= 120^\circ$



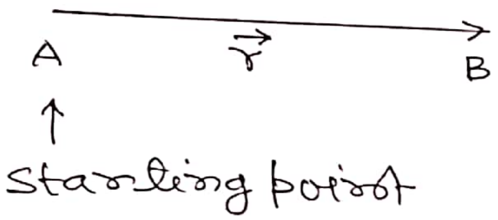
Type of Vectors

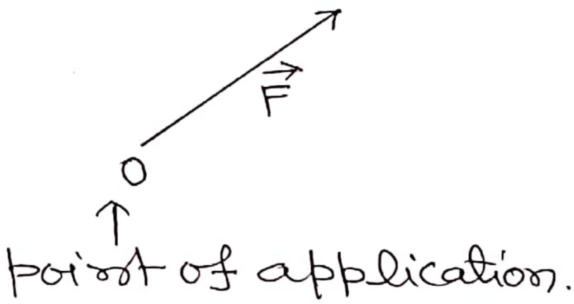
Mainly vectors are of two types

(1) Polar Vector (2) Axial Vector / Pseudo Vector

Polar Vector – It is a vector which has either starting point or point of application and independent of the orientation of an axis.

Ex – displacement, force, momentum, velocity, acceleration etc.

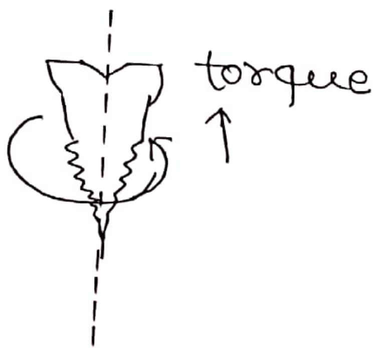


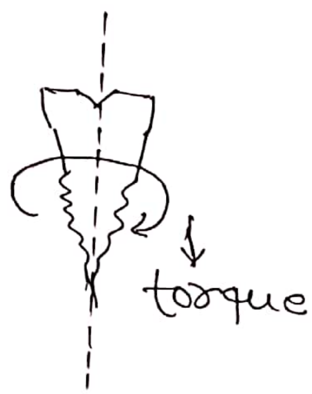


Axial Vector – It is a vector which depends on the orientation of an axis and is along the axis.

Ex – Torque, angular displacement, angular velocity, angular acceleration, angular momentum etc.

⇒ An axial vector also depends on the sense of rotation / movement.





Common type of vectors -

Proper Vector - It is a vector which magnitude is nonzero.

Let $|\vec{A}| \neq 0$, then $\vec{A}$ is a proper vector.

Zero vector - It is a vector which magnitude is zero.

Let $|\vec{A}| = 0$, then $\vec{A}$ is a zero vector.

⇒ A zero vector is the resultant of two or more than two vectors.

⇒ The direction of a zero vector is indermine but is specific.

⇒ The direction of zero vector can be taken in any direction.

⇒ Let $\vec{0}$ = zero vector, then

$$\vec{A} + \vec{0} = \vec{A}$$
$$\vec{A} - \vec{0} = \vec{A}$$
$$\vec{A} \cdot \vec{0} = 0$$
$$\vec{A} \times \vec{0} = \vec{0}$$

Unit vector - It is a vector which magnitude is equal to 1 (one).

⇒ Unit vector $\vec{A}$ is also denoted as $\hat{A}$.

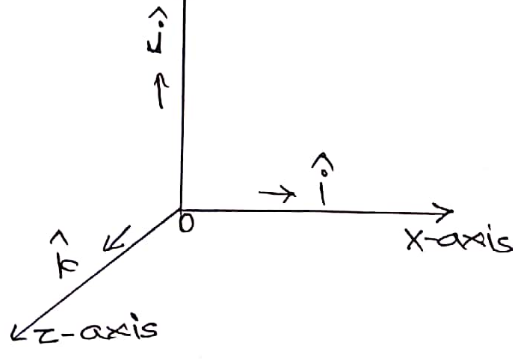
⇒ To find a unit vector along a vector, the vector is divided by its magnitude.

ex- a unit vector along $\vec{A}$

$$\hat{u} = \frac{\vec{A}}{|\vec{A}|}$$

⇒ A unit vector is unitless.

⇒ Conventionally unit vector along three perpendicular co-ordinate directions x-axis, y-axis and z-axis are denoted by $\hat{i}$, $\hat{j}$ and $\hat{k}$ respectively.



⇒ A unit vector is used to represent the direction of a vector.

ex- When velocity $\vec{V} = 10\, m s^{-1}$ in x-axis
then $\vec{V} = (10\, m s^{-1})\, \hat{i}$

When acceleration $\vec{a} = 5\, m s^{-2}$ in Y-axis
then $\vec{a} = (5\, m s^{-2})\, \hat{j}$

When force $\vec{F} = 8N$ in x-axis and $6N$ in Y-axis
then $\vec{F} = (8N)\hat{i} + (6N)\hat{j}$

Parallel Vectors- Two vectors which directions are same are known as parallel vectors.

ex- $\vec{A} = 6m\,\hat{i}$, $\vec{B} = 8m\,\hat{i}$

⇒ Angle between two parallel vectors is 0°.

⇒ Parallel vectors may be of different dimensional formula.

⇒ Parallel vectors may have different magnitude.

Antiparallel Vectors- Two vectors which directions are just opposite are known as antiparallel vectors.

ex- $\vec{A} = 6m\,\hat{i}$, $\vec{B} = -8\hat{i}$

⇒ Angle between antiparallel vectors is 180°.

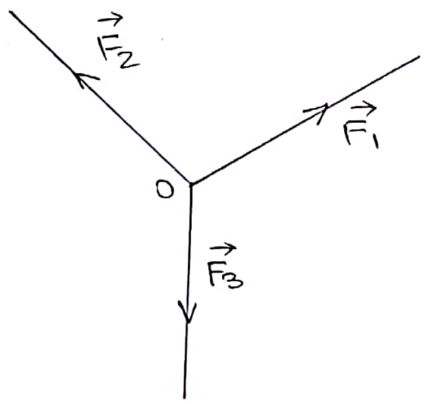
Coaxial Vectors / Colinear Vectors - Parallel vectors and antiparallel vectors in combined are known as coaxial vectors.

ex- $\vec{A} = 2m\,\hat{i}$, $\vec{B} = 4m\,\hat{i}$, $\vec{C} = -5m\,\hat{i}$

⇒ Angle between coaxial vectors may 0° or may 180°.

Coinitial Vectors - Vectors which initial points are common are known as coinitial vectors.

In figure $\vec{F_1}$, $\vec{F_2}$ and $\vec{F_3}$ are coinitial forces.



Coplaner vectors – Vectors which exist in same plane are known as coplaner vectors.

ex- $\vec{A} = 2m\hat{i} + 5m\hat{j}$, $\vec{B} = 4m\hat{i} + 3m\hat{j}$

Equal Vectors – Two vectors which magnitude and direction are identical are called equal vectors.

ex- $\vec{A} = 4m\hat{i} + 6m\hat{j}$, $\vec{B} = 4m\hat{i} + 6m\hat{j}$

⇒ Equal vectors must be parallel vectors

⇒ Difference of equal vectors is a zero vector.

Opposite vectors/ Negative vectors – Two vectors which magnitude are equal but the directions are just opposite are known as opposite vectors.

ex- $\vec{P} = 2m\hat{i} + 3m\hat{j}$, $\vec{Q} = -2m\hat{i} - 3m\hat{j}$

⇒ Angle between opposite vectors is $180°$.

⇒ Opposite vectors are antiparallel.

⇒ Vector sum of opposite vectors is a zero vector.

Free vector – a vector which initial point can be taken as any point is known as free vector.

ex- $\vec{S} = 10\ m$ along east

Localised Vector – a vector which initial point is localised is called localised vector.

ex- displacement

$\vec{S} = 10\ m$ along east from origin.

Practice problems

(1) Which of the following represents a vector

(A) $\longleftrightarrow$ (B) $\longrightarrow$ (C) $\overleftarrow{A}$ (D) $\vec{A}$

(2) Let $\hat{u}$ = a unit vector along $\vec{A}$ then a correct relation is

(A) $\vec{A} = \vec{A}\,u$ (B) $\vec{A} = |\vec{A}|\,u$

(C) $\vec{A} = |\vec{A}|\,\hat{u}$ (D) None of these

(3) A unit vector along x-axis is

(A) $\hat{i}$ (B) $\hat{j}$ (C) $\hat{k}$ (D) $-\hat{i}$

(4) Which one is the correct representation of a force of 6N in Y-axis

(A) $(6\,N)\hat{i}$ (B) $(6\,N)\hat{j}$ (C) $(6N)\hat{k}$ (D) $(-6\,N)\hat{i}$

(5) correct representation of a velocity of $10\,ms^{-1}$ in negative x-axis is

(A) $(10\,ms^{-1})\hat{i}$ (B) $(-10\,ms^{-1})\hat{i}$ (C) $(10\,ms^{-1})\hat{j}$ (D) $(10\,ms^{-1})\hat{k}$

(6) Let $|\vec{A}| = 1$, then $\vec{A}$ is a

(A) zero vector (B) unit vector

(C) Axial vector (D) None of these

(7) Let $\vec{A} + \vec{B} = \vec{0}$ (zero vector), then $\vec{A}$ and $\vec{B}$ are

(A) Equal vectors (B) Opposite vectors

(C) Parallel vectors (D) Antiparallel vectors

(8) Let $\vec{A} - \vec{B} = \vec{0}$ (zero vector), then $\vec{A}$ and $\vec{B}$ are

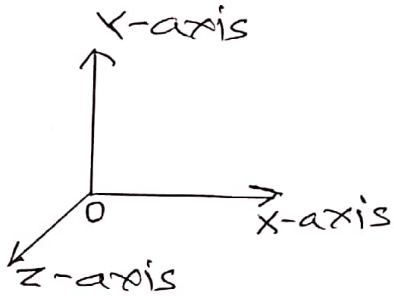
(A) Equal vectors (B) Opposite vectors

(C) Parallel vectors (D) Antiparallel vectors

Ans- (1) - B, D (2) - C (3) - A (4) - B
(5) - B (6) - B (7) - B, D (8) - A, C

Comprehension

Question number (9)-(12) are based on the statements given below



$\vec{A} = (2\text{ units})\,\hat{i}$

$\vec{B} = (3\text{ units})\,\hat{j}$

$\hat{i}$, $\hat{j}$ and $\hat{k}$ represents unit vectors along x-axis, Y-axis and z-axis respectively.

(9) Geometrical representation of $\vec{A}$ is

(A) $\xrightarrow[2\text{ units}]{}$ (B) $\uparrow$ 2 units (C) $\xleftarrow[2\text{ units}]{}$ (D) $\overline{}$ 2 units

(10) Geometrical representation of $\vec{B}$ is

(A) $\xrightarrow[3\text{ units}]{}$ (B) $\uparrow$ 3 units (C) $\xleftarrow[3\text{ units}]{}$ (D) $\nearrow$ 3 units

(11) Geometrical representation of $-4\vec{A}$ is

(A) $\xleftarrow[8\text{ units}]{}$ (B) $\xrightarrow[8\text{ units}]{}$ (C) $\uparrow$ 8 units (D) $\downarrow$ 8 units

(12) Geometrical representation of $3\vec{B}$ is

(A) $\xrightarrow[9\text{ units}]{}$ (B) $\xleftarrow[9\text{ units}]{}$ (C) $\uparrow$ 9 units (D) $\downarrow$ 9 units

Ans- (9) – A (10) – B

(11) – A (12) – C

Addition of vectors

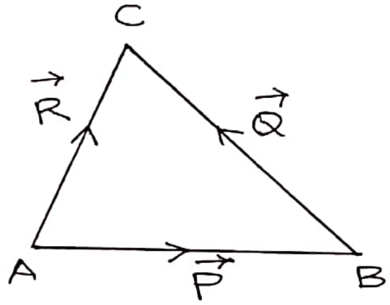
Two or more than two vectors are added by some specific rules known as rules of addition of Vectors. Rules are as follows

(A) Triangle rule of addition of Vectors – When two vectors are geometrically represented by two sides of a triangle in same order then the third side of the triangle in opposite order represents the vector sum both in magnitude and direction.

In triangle ABC

$\vec{AB} + \vec{BC} = \vec{AC}$

$\Rightarrow \vec{R} = \vec{P} + \vec{Q}$



According to triangle law –

$\vec{AB} + \vec{BC} = \vec{AC}$

or

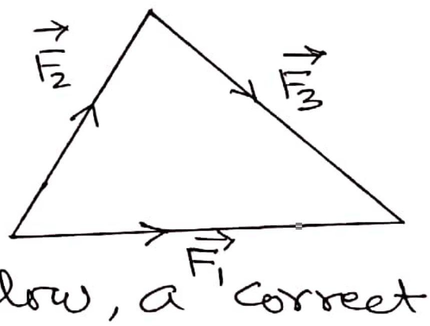
$\vec{AC} + \vec{CB} = \vec{AB}$

Ex (1) – In the figure given below, a correct relation is

(A) $\vec{F_1} + \vec{F_2} = \vec{F_3}$ (B) $\vec{F_2} + \vec{F_3} = \vec{F_1}$

(C) $\vec{F_3} + \vec{F_1} = \vec{F_2}$ (D) $\vec{F_1} + \vec{F_2} + \vec{F_3} = 0$

Ans – (B)

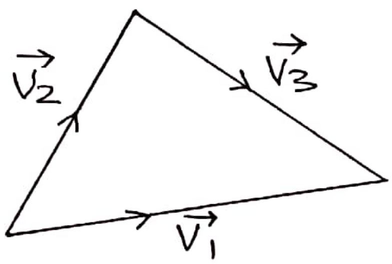


Ex (2) – In the figure given below, a correct relation is

(A) $\vec{V_1} - \vec{V_2} = \vec{V_3}$ (B) $\vec{V_2} - \vec{V_3} = \vec{V_1}$

(C) $\vec{V_3} - \vec{V_1} = \vec{V_2}$ (D) $\vec{V_2} - \vec{V_1} = \vec{V_3}$

Ans – (A)

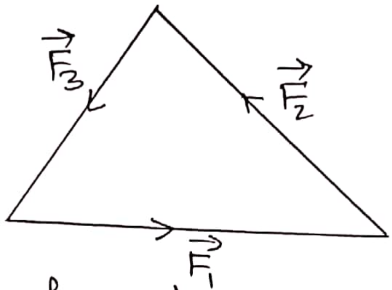


Ex(3) – In the figure given below a correct relation between forces $\vec{F_1}$, $\vec{F_2}$ and $\vec{F_3}$ is

(A) $\vec{F_1} + \vec{F_2} = \vec{F_3}$ (B) $\vec{F_2} + \vec{F_3} = \vec{F_1}$

(C) $\vec{F_3} + \vec{F_1} = \vec{F_2}$ (D) $\vec{F_1} + \vec{F_2} + \vec{F_3} = 0$

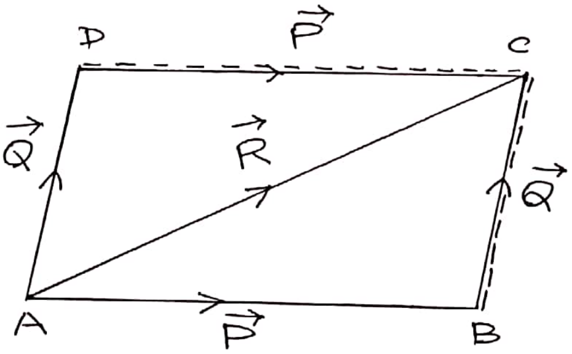
Ans – (D)



(B) Parallelogram law of addition of vectors –

When initial point of two vectors make coincide and a parallelogram is drawn by taking two vectors as two adjacent sides then a ~~po~~ diagonal passing through coinciding tails represents the vector sum both in magnitude and direction.

In figure.



$\vec{AB} = \vec{P}$, $\vec{AD} = \vec{Q}$

$\vec{AC} = \vec{R}$

a/c to parallelogram rule -

$\vec{AB} + \vec{AD} = \vec{AC}$

$\therefore \vec{P} + \vec{Q} = \vec{R}$

$\Rightarrow$ In parallelogram rule, triangle rule is followed.

as $\vec{AB} + \vec{BC} = \vec{AC}$

$\Rightarrow$ Vectors addition is commutative

In Δ ABC	In Δ ADC
$\vec{AB} + \vec{BC} = \vec{AC}$	$\vec{AD} + \vec{DC} = \vec{AC}$
$\Rightarrow \vec{P} + \vec{Q} = \vec{R}$	$\vec{Q} + \vec{P} = \vec{R}$

So $\vec{P} + \vec{Q} = \vec{Q} + \vec{P}$

(C) Polygon rule of addition of vectors – to add more than two vectors polygon rule is followed. When more than two vectors are drawn as initial point of 2nd vector is at terminal point of 1st vector, initial point of 3rd vector is at terminal point of 2nd vector and so on, then a line-segment starting from initial point of 1st vector to the terminal point of last vector represents the vector sum of all vectors.

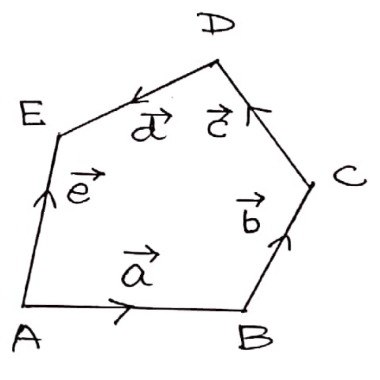
In figure

$\vec{AB} + \vec{BC} + \vec{CD} + \vec{DE} = \vec{AE}$

$\Rightarrow \vec{a} + \vec{b} + \vec{c} + \vec{d} = \vec{e}$

$\vec{AB} + \vec{BC} + \vec{CD} + \vec{DE} = -\vec{EA}$

$\Rightarrow \vec{AB} + \vec{BC} + \vec{CD} + \vec{DE} + \vec{AE} = 0$



Triangle law and polygon law

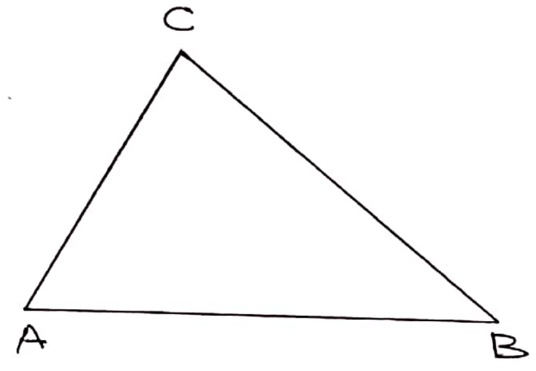
A/c to polygon law

$\vec{AB} + \vec{BC} + \vec{CA} = 0$

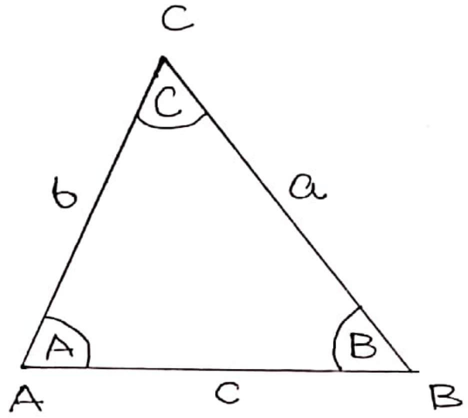
$\Rightarrow \vec{AB} + \vec{BC} = -\vec{CA}$

$\Rightarrow \vec{AB} + \vec{BC} = \vec{AC}$

This is triangle rule.



Two important laws of triangle



According to sine law

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

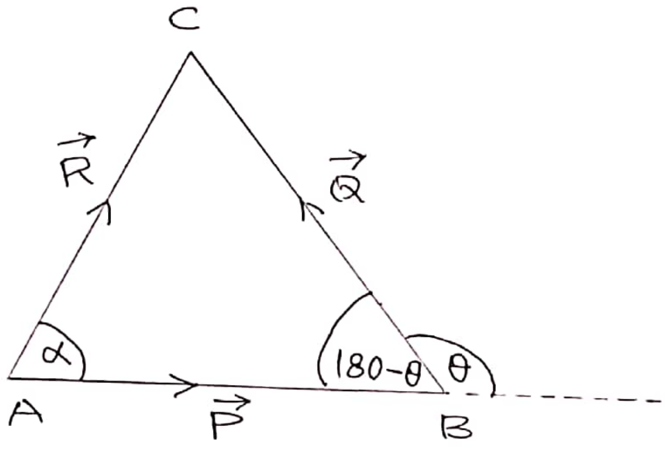
According to cosine law

$$a = \sqrt{b^2 + c^2 - 2bc\cos A}$$

$$b = \sqrt{c^2 + a^2 - 2ca\cos B}$$

$$c = \sqrt{a^2 + b^2 - 2ab\cos C}$$

Expression of magnitude and direction of resultant vector sum -



In figure

$\vec{AB} = \vec{P} \Rightarrow |\vec{P}| = AB$

$\vec{BC} = \vec{Q} \Rightarrow |\vec{Q}| = BC$

$\vec{AC} = \vec{R} \Rightarrow |\vec{R}| = AC$

According to triangle rule -

$$\vec{AC} = \vec{AB} + \vec{BC}$$

$$\therefore \vec{R} = \vec{P} + \vec{Q}$$

Let angle between $\vec{P}$ and $\vec{Q}$ is θ, then a/c to cosine law

$$AC = \sqrt{(AB)^2 + (BC)^2 - 2(AB)(BC)\cos(180-\theta)}$$

$$= \sqrt{(AB)^2 + (BC)^2 + 2(AB)(BC)\cos\theta}$$

$$\therefore |\vec{R}| = \sqrt{|\vec{P}|^2 + |\vec{Q}|^2 + 2|\vec{P}||\vec{Q}|\cos\theta}$$

So in magnitude

$$|\vec{R}| = \sqrt{|\vec{P}|^2 + |\vec{Q}|^2 + 2|\vec{P}||\vec{Q}|\cos\theta} \quad \text{— (A)}$$

Let $\vec{R}$ makes an angle of α with $\vec{P}$, then a/c to sine law

$$\frac{BC}{\sin\alpha} = \frac{AC}{\sin(180-\theta)}$$

$$\Rightarrow \sin\alpha = \frac{BC\sin\theta}{AC}$$

$$\therefore \sin\alpha = \frac{|\vec{Q}|\sin\theta}{|\vec{R}|} \quad \text{— (B)}$$

Also

$$\tan\alpha = \frac{|\vec{Q}|\sin\theta}{|\vec{P}|+|\vec{Q}|\cos\theta} \quad \text{— (C)}$$

Above relation (A) gives magnitude where relation (B) & (C) gives direction.

Special cases -

(1) When $\theta = 0^\circ$, $\cos 0^\circ = 1$ (max^m value), then

$$|\vec{R}| = \sqrt{|\vec{P}|^2 + |\vec{Q}|^2 + 2|\vec{P}||\vec{Q}|(1)}$$

$$\therefore |\vec{R}| = |\vec{P}| + |\vec{Q}| \quad \text{max}^m \text{ value}$$

And $\alpha = 0^\circ$

(2) When $\theta = 180^\circ$, $\cos 180^\circ = -1$ (min. value), then

$$|\vec{R}| = |\vec{P}| - |\vec{Q}| \text{ or } |\vec{Q}| - |\vec{P}|$$

And $\alpha = 0^\circ$ or 180°

(3) When $\theta = 90^\circ$, $\cos 90^\circ = 0$, then

$$|\vec{R}| = \sqrt{|\vec{P}|^2 + |\vec{Q}|^2}$$

and $\tan\alpha = \dfrac{|\vec{Q}|}{|\vec{P}|}$

(4) When $|\vec{P}| = |\vec{Q}|$, then

$$|\vec{R}| = 2|\vec{P}|\cos\left(\frac{\theta}{2}\right)$$

and $\alpha = \frac{\theta}{2}$

(5) When $|\vec{P}| = |\vec{Q}|$ and $\theta = 90^\circ$, then

$|\vec{R}| = |\vec{P}|\sqrt{2}$ and $\alpha = 45^\circ$

(6) When $|\vec{P}| = |\vec{Q}|$ and $\theta = 120^\circ$, then

$|\vec{R}| = |\vec{P}| = |\vec{Q}|$ and $\alpha = 60^\circ$

Ex(1) - Magnitude of two coinitial forces are 8N and 6N. If angle between them is 0°, then magnitude of resultant force will -

(A) 10N (B) 14N (C) 2N (D) 12N

Ans - (B) Solⁿ $\vec{F} = \vec{F_1} + \vec{F_2}$

as $\theta = 0^\circ$

$\therefore |\vec{F}| = |\vec{F_1}| + |\vec{F_2}| = 8N + 6N = 14N$

Ex(2) - Two coinitial forces are 8N and 6N in magnitude. If angle between them is 180°, then magnitude of resultant force will -

(A) 2N (B) 10N (C) 14N (D) 16N

Ans - (A) Solⁿ at $\theta = 180^\circ$

$|\vec{F}| = |\vec{F_1}| - |\vec{F_2}| = 8N - 6N = 2N$

Ex(3) - Two coinitial forces of magnitude 8N and 6N are at 90°. Magnitude of resultant force will -

(A) 2N (B) 10N (C) 14N (D) 16N

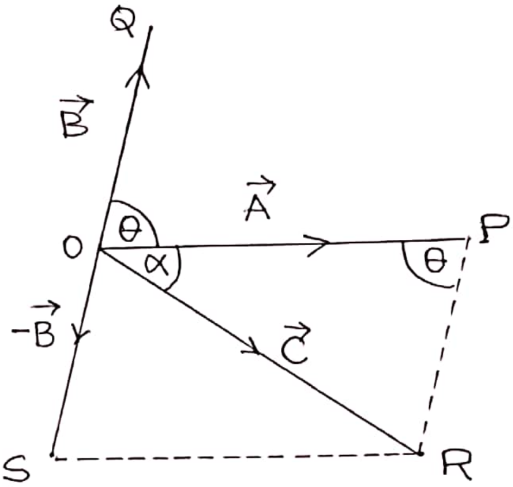
Ans - (B) Solⁿ at $\theta = 90^\circ$

$|\vec{F}| = \sqrt{|\vec{F_1}|^2 + |\vec{F_2}|^2} = \sqrt{8^2 + 6^2}\,N = 10N$

Subtraction of vectors

Let
$$\vec{C} = \vec{A} - \vec{B}$$
$$= \vec{A} + (-\vec{B})$$

Thus to subtract $\vec{B}$ from $\vec{A}$ we should add negative of $\vec{B}$ to $\vec{A}$.



In figure –

$\vec{OP} = \vec{A} \Rightarrow |\vec{A}| = OP$

$\vec{OQ} = \vec{B}$

$\vec{OS} = -\vec{B} \Rightarrow |\vec{B}| = OQ = OS = PR$

Angle between $\vec{A}$ & $\vec{B}$ is θ

Angle between $\vec{C}$ & $\vec{A}$ is α

According to triangle law of addition
$$\vec{OR} = \vec{OP} + \vec{PR}$$
$$\therefore \vec{C} = \vec{A} - \vec{B}$$

In $\triangle$ OPR a/c to cosine law
$$OR = \sqrt{(OP)^2 + (PR)^2 - 2(OP)(PR)\cos\theta}$$
$$\therefore |\vec{C}| = \sqrt{|\vec{A}|^2 + |\vec{B}|^2 - 2|\vec{A}||\vec{B}|\cos\theta} \quad \text{——— (A)}$$

In $\triangle$ OPR a/c to sine law
$$\frac{PR}{\sin\alpha} = \frac{OR}{\sin\theta} \Rightarrow \frac{|\vec{B}|}{\sin\alpha} = \frac{|\vec{C}|}{\sin\theta}$$
$$\therefore \sin\alpha = \frac{|\vec{B}|\sin\theta}{|\vec{C}|} \quad \text{——— (B)}$$

Also
$$\tan\alpha = \frac{|\vec{B}|\sin\theta}{|\vec{A}| - |\vec{B}|\cos\theta} \quad \text{——— (C)}$$

Above relation (A) is used to find magnitude and relations (B) & (C) are used to find direction of resultant vector $\vec{C}$.

Special cases -

(1) When $\theta = 0^\circ$, $\cos 0^\circ = 1$, then

$|\vec{C}| = |\vec{A}| - |\vec{B}|$ if $|\vec{A}| > |\vec{B}|$
and $\alpha = 0^\circ$

OR

$|\vec{C}| = |\vec{B}| - |\vec{A}|$ if $|\vec{A}| < |\vec{B}|$
and $\alpha = 180^\circ$

(2) When $\theta = 180^\circ$, $\cos 180^\circ = -1$, then

$|\vec{C}| = |\vec{A}| + |\vec{B}|$ and $\alpha = 180^\circ$

(3) When $\theta = 90^\circ$, $\cos 90^\circ = 0$, then

$|\vec{C}| = \sqrt{|\vec{A}|^2 + |\vec{B}|^2}$

and $\tan \alpha = \frac{|\vec{B}|}{|\vec{A}|}$

(4) When $|\vec{A}| = |\vec{B}|$, then

$|\vec{C}| = 2|\vec{A}| \sin\left(\frac{\theta}{2}\right)$

and $\alpha = 90^\circ - \left(\frac{\theta}{2}\right)$

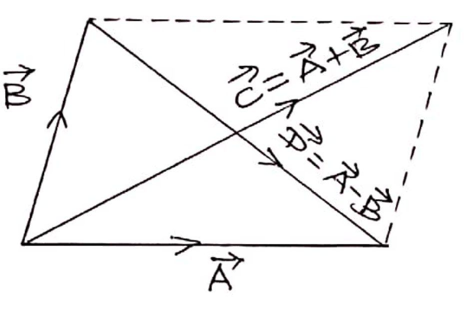
(5) When $|\vec{A}| = |\vec{B}|$ and $\theta = 90^\circ$ then

$|\vec{C}| = |\vec{A}|\sqrt{2}$

(6) When $|\vec{A}| = |\vec{B}|$ and $\theta = 60^\circ$ then

$|\vec{C}| = |\vec{A}| = |\vec{B}|$

⇒ Let initial point of two vectors coincide and a parallelogram is prepared by taking two vectors as two adjacent sides then where a diagonal gives resultant sum the another diagonal gives resultant subtraction.



Ex(1) – A car is moving with velocity 8 m/s in east. At a turning of road it turns towards north and it velocity becomes 6 m/s. To find change in velocity of the car –

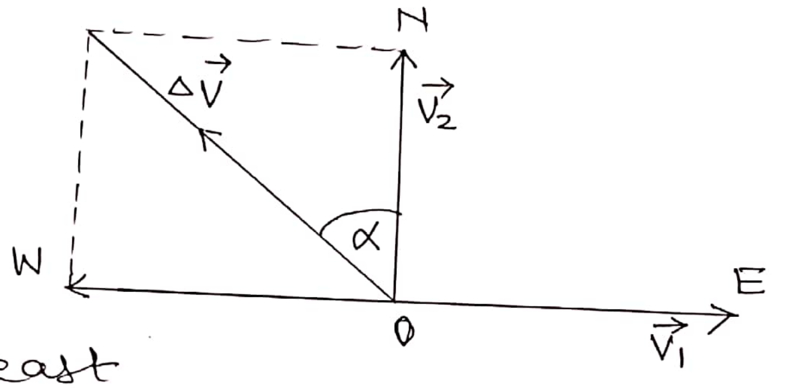
(A) 2 m/s in east (B) 2 m/s in north

(C) 10 m/s at 53° West of north

(D) 10 m/s at 53° north of west

Ans (C)

Soln.



$\vec{V_1} = 8$ m/s in east

$\vec{V_2} = 6$ m/s in north

Change in Velocity

$$\Delta\vec{V} = \vec{V_2} - \vec{V_1}$$

$$\theta = 90^\circ$$

$$\therefore |\Delta\vec{V}| = \sqrt{|\vec{V_2}|^2 + |\vec{V_1}|^2}$$

$$= \sqrt{6^2 + 8^2} \text{ m/s}$$

$$= \sqrt{36+64} \text{ m/s}$$

$$= \sqrt{100} \text{ m/s}$$

$$= 10 \text{ m/s}$$

Direction

$$\tan\alpha = \frac{|\vec{V_1}|}{|\vec{V_2}|}$$

$$= \frac{8}{6}$$

$$= \frac{4}{3}$$

$$\alpha = \tan^{-1}(4/3) = 53^\circ$$

Ex(2) – Momentum of a block is 12 Ns in opposite of x-axis. Under influence of a force its momentum becomes 16 Ns in Y-axis in 4.0 s. To find average force on the block.

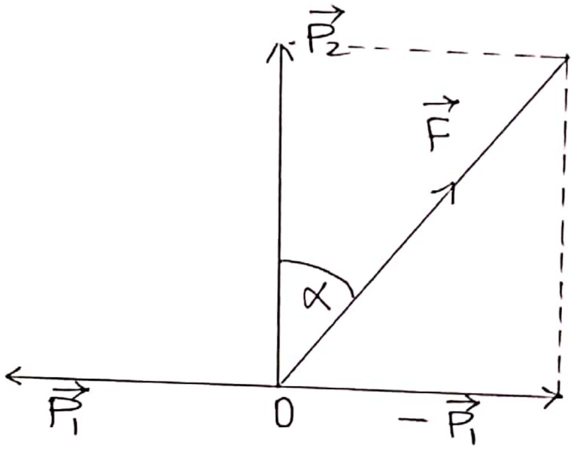
(A) 5.0 N in x-axis (B) 6.0 N in Y-axis

(C) 5.0 N at 37° with Y-axis

(D) 6.0 N at 37° with Y-axis.

Ans – (C)

Sol^n



$\vec{P_1} = -12$ Ns in x-axis

$\vec{P_2} = 16$ Ns in Y-axis

$\Delta t = 4.0$ s

change in momentum

$$\Delta\vec{P} = \vec{P_2} - \vec{P_1}$$

$$\theta = 90^\circ$$

$$|\Delta\vec{P}| = \sqrt{|\vec{P_2}|^2 + |\vec{P_1}|^2}$$

$$= \sqrt{(16)^2 + (12)^2} \text{ Ns}$$

$$= \sqrt{256 + 144} \text{ Ns}$$

$$= \sqrt{400} \text{ Ns}$$

$$= 20 \text{ Ns}$$

Average force

$$F = \frac{\Delta P}{\Delta t} = \frac{20 \text{ Ns}}{4.0 \text{ s}} = 5.0 \text{ N}$$

Direction

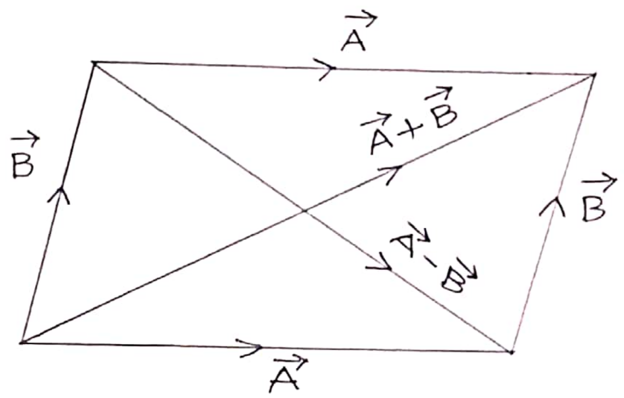
$$\tan\alpha = \frac{|\vec{P_1}|}{|\vec{P_2}|} = \frac{12}{16}$$

$$= \frac{3}{4}$$

$$\alpha = \tan^{-1}(3/4) = 37^\circ$$

Important points of addition and subtraction

(1) When initial point of two vectors are taken at same point and a parallelogram is drawn by taking two vectors as two adjacent sides then where a diagonal represents vectors sum the another diagonal represents vectors subtraction.



(2) Vectors addition is commutative

$$\vec{A}+\vec{B} = \vec{B}+\vec{A}$$

(3) Vectors subtraction is not commutative

$$\vec{A}-\vec{B} \neq \vec{B}-\vec{A}$$

(4) Let $\vec{C} = \vec{A}+\vec{B}$ then

$$|\vec{A}|-|\vec{B}| \leq |\vec{C}| \leq |\vec{A}|+|\vec{B}|$$

$$180^\circ \geq \theta \geq 0^\circ$$

(5) Let $\vec{C} = \vec{A}-\vec{B}$ then

$$|\vec{A}|-|\vec{B}| \leq |\vec{C}| \leq |\vec{A}|+|\vec{B}|$$

$$0^\circ \leq \theta \leq 180^\circ$$

(6) When $\vec{A}+\vec{B}$ then

$$\vec{A} = -\vec{B}$$

(7) When $\vec{A}+\vec{B}+\vec{C}=0$, then

$\vec{A}+\vec{B}=-\vec{C}$, $\vec{B}+\vec{C}=-\vec{A}$, $\vec{C}+\vec{A}=-\vec{B}$.

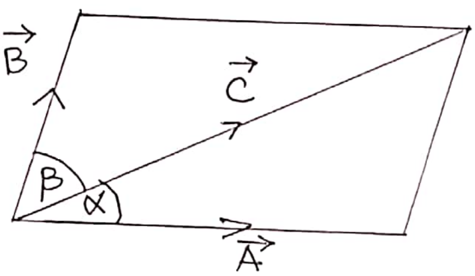
$|\vec{A}|-|\vec{B}| \leq |\vec{C}| \leq |\vec{A}|+|\vec{B}|$

Clearly sum of three vectors may be zero only when magnitude of biggest vector is either small or equal to the sum of magnitude of rest vectors.

(8) Let $\vec{C}=\vec{A}+\vec{B}$

α = angle between $\vec{C}$ & $\vec{A}$

β = angle between $\vec{C}$ & $\vec{B}$



Then

(i) when $|\vec{A}|>|\vec{B}|$ then $\alpha<\beta$

(ii) when $|\vec{A}|<|\vec{B}|$ then $\alpha>\beta$

(iii) when $|\vec{A}|=|\vec{B}|$ then $\alpha=\beta$

⇒ Minimum number of vectors of equal magnitude which addition is zero is '2'.

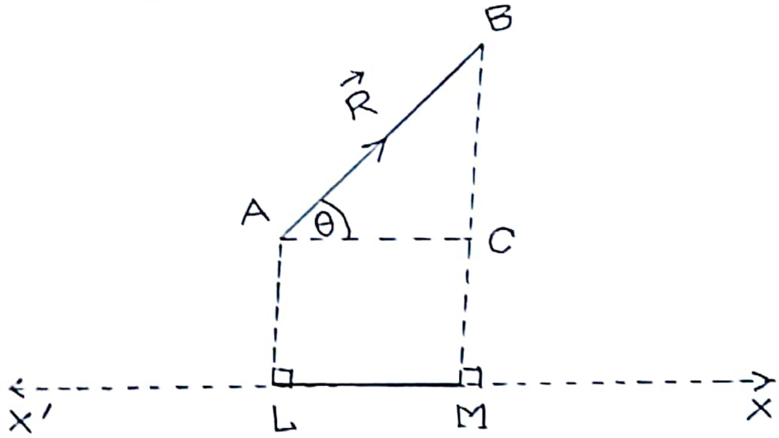
⇒ Minimum number of vectors of unequal magnitude which addition is zero is '3'.

⇒ Minimum number of vectors of noncoplaner and equal magnitude to make resultant sum zero is '3'.

⇒ Minimum number of noncoplaner vectors of unequal magnitude to make resultant sum zero is '4'.

Resolution of vector and its component

Component – When perpendiculars are drawn from both initial point and terminal point of a vector on a line/axis, then smallest distance between two perpendiculars is known as component of the vector along that line/axis.



In figure

LM = component of $\vec{R}$ along line $x'x$

= resolution part of $\vec{R}$ along line $x'x$

= projection part of $\vec{R}$ along line $x'x$

Measurement of component –

Let $\vec{R}$ makes an angle of θ with line $x'x$

In $\triangle ACB$

$$\frac{AC}{AB} = \cos\theta$$

$\Rightarrow AC = AB\cos\theta$

$\Rightarrow LM = AB\cos\theta$

$\Rightarrow LM = |\vec{R}|\cos\theta$

Thus component of $\vec{R}$ along a direction at an angle θ

$= |\vec{R}|\cos\theta$

⇒ component of a vector is a scalar.

* Components of $\vec{R}$ along x-axis, Y-axis and Z-axis are denoted by R_x, R_y and R_z respectively

Ex(1) – To find component of a force of magnitude 20N in a direction at an angle of 30°.

(A) 10N (B) $10\sqrt{3}$ N (C) 15 N (D) 20N

Ans – (B)

Sol^n $|\vec{F}| = 20$ N , $\theta = 30°$

component $= |\vec{F}| \cos\theta$

$= (20N)\cos 30° = (20N)(\sqrt{3}/2) = 10\sqrt{3}$ N

Ex(2) – Velocity of a particle is $15\ m\ s^{-1}$ at an angle of 37° with x-axis in x-Y plane. To find x-component and Y-component of the velocity.

(A) $V_x = 9\ m\ s^{-1}$, $V_y = 12\ m\ s^{-1}$

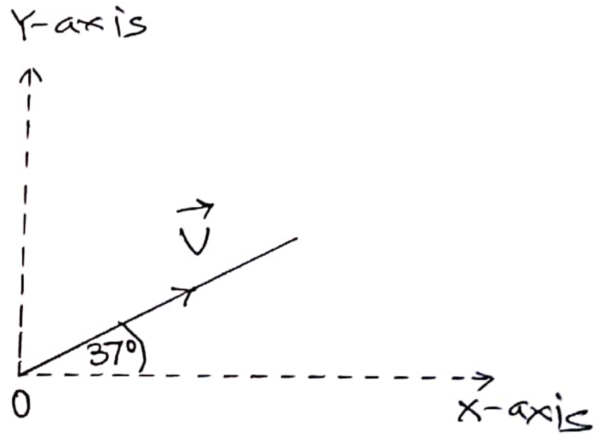
(B) $V_x = 10\ m\ s^{-1}$, $V_y = 5\ m\ s^{-1}$

(C) $V_x = 12\ m\ s^{-1}$, $V_y = 9\ m\ s^{-1}$

(D) $V_x = V_y = 7.5\ m\ s^{-1}$

Ans– (C)

Sol^n $|\vec{V}| = 15\ m\ s^{-1}$



angle with x-axis

$\alpha = 37°$

$\therefore V_x = |\vec{V}|\cos\alpha$

$= (15\ m\ s^{-1})\cos 37°$

$= (15\ m\ s^{-1}) \times (\frac{4}{5})$

$= 12\ m\ s^{-1}$

angle with y-axis

$\beta = 90° - 37° = 53°$

$V_y = |\vec{V}|\cos\beta = (15\ m\ s^{-1})\cos 53°$

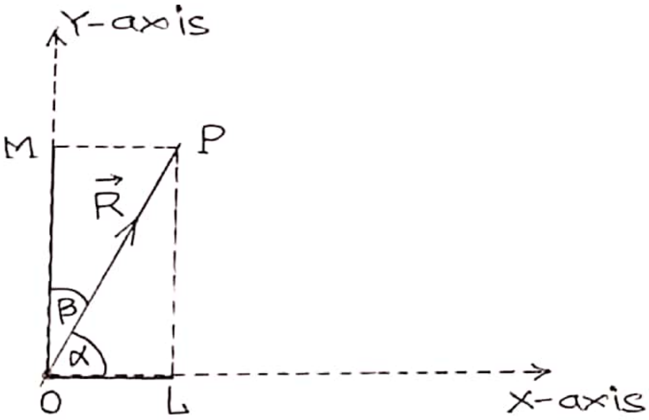
$= (15\ m\ s^{-1})(3/5)$

$= 9\ m\ s^{-1}$.

Resolution of vector - It is a process of breaking a vector into its components.

Resolution in plane

Suppose a vector $\vec{R} = \vec{OP}$ in x-y plane which makes an angle α with x-axis and an angle of β with y-axis.



Construction -

PL ⊥ X-axis
& PM ⊥ Y-axis.

x-component of $\vec{R}$

$R_x = OL = |\vec{R}|\cos\alpha$

y-component of $\vec{R}$

$R_y = OM = |\vec{R}|\cos\beta$

$= |\vec{R}|\cos(90-\alpha)$

$= |\vec{R}|\sin\alpha$

Component form of $\vec{R}$

According to parallelogram law in OLPM

$$\vec{OP} = \vec{OL} + \vec{OM}$$

$$\vec{OP} = (OL)\hat{i} + (OM)\hat{j}$$

$$\therefore \boxed{\vec{R} = R_x\hat{i} + R_y\hat{j}}$$

Also

$$\boxed{\vec{R} = |\vec{R}|\cos\alpha\,\hat{i} + |\vec{R}|\sin\alpha\,\hat{j}}$$

⇒ A unit vector in the direction of $\vec{R}$

$$\hat{u} = \frac{\vec{R}}{|\vec{R}|} = \cos\alpha\,\hat{i} + \sin\alpha\,\hat{j}$$

Magnitude of $\vec{R}$

In rectangle OLPM

$$OP = \sqrt{(OL)^2 + (OM)^2}$$

$$\therefore \boxed{|\vec{R}| = \sqrt{R_x^2 + R_y^2}}$$

Direction of $\vec{R}$ with x-axis

$$\cos\alpha = \frac{R_x}{|\vec{R}|} \Rightarrow \alpha = \cos^{-1}\left(\frac{R_x}{|\vec{R}|}\right)$$

$$\tan\alpha = \frac{R_y}{R_x} \Rightarrow \alpha = \cos^{-1}\left(\frac{R_y}{R_x}\right)$$

Direction of $\vec{R}$ with y-axis

$$\cos\beta = \frac{R_y}{|\vec{R}|} \Rightarrow \beta = \cos^{-1}\left(\frac{R_y}{|\vec{R}|}\right)$$

$$\tan\beta = \frac{R_x}{R_y} \Rightarrow \beta = \tan^{-1}\left(\frac{R_x}{R_y}\right)$$

Ex(1) - To find magnitude of $\vec{A} = 6m\hat{i} + 8m\hat{j}$

(A) 14m (B) 10m (C) 2m (D) 16m

Ans - (B), Sol^n $\vec{A} = 6m\hat{i} + 8m\hat{j}$

as $\vec{A} = A_x\hat{i} + A_y\hat{j}$

$\Rightarrow A_x = 6m, A_y = 8m$

$$\therefore |\vec{A}| = \sqrt{A_x^2 + A_y^2} = \sqrt{6^2 + 8^2} = 10m$$

Ex(2) - Angle that $\vec{A} = 6m\hat{i} + 8m\hat{j}$ makes with x-axis is

(A) 53° (B) 37° (C) 30° (D) 60°

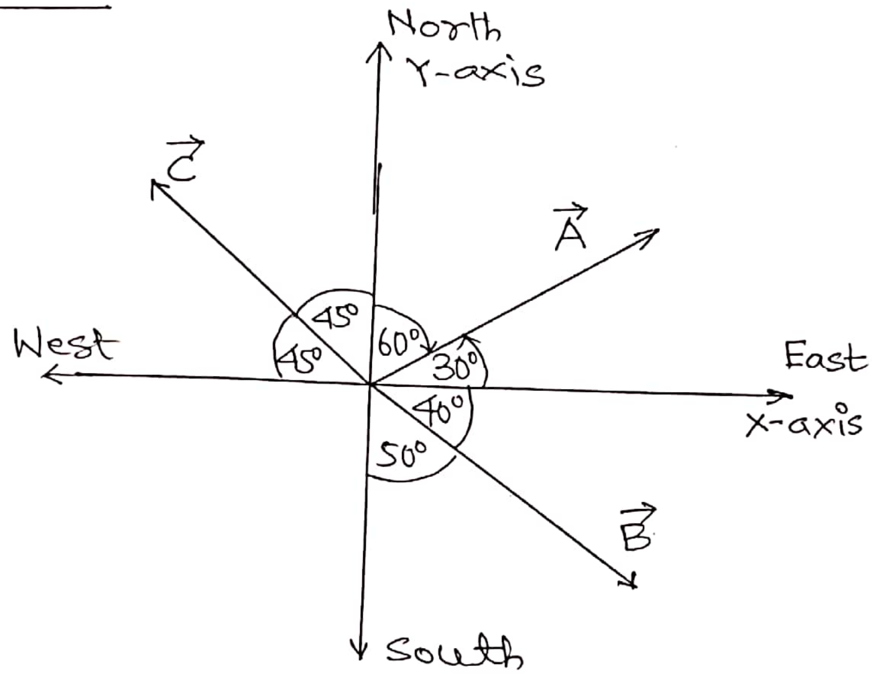
Ans - (B) Sol^n $\vec{A} = 6m\hat{i} + 8m\hat{j}$

Angle with x-axis

$$\alpha = \tan^{-1}\left(\frac{A_y}{A_x}\right) = \tan^{-1}(8/6)$$

$$= \tan^{-1}(4/3) = 37^\circ$$

Direction convention



$\vec{A}$ is at 30° north of east

$\vec{A}$ is at 60° east of north

$\vec{B}$ is at 40° south of east

$\vec{B}$ is at 50° east of south

$\vec{C}$ is along north-west

$\vec{C}$ is along west-north.

Angle convention

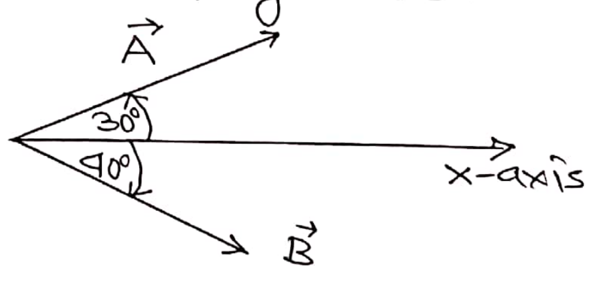
When angle of number of vectors are taken with an axis. If an angle in anticlockwise is taken as positive, then an angle in clockwise sense is taken as negative.

Angle of $\vec{A}$ with x-axis

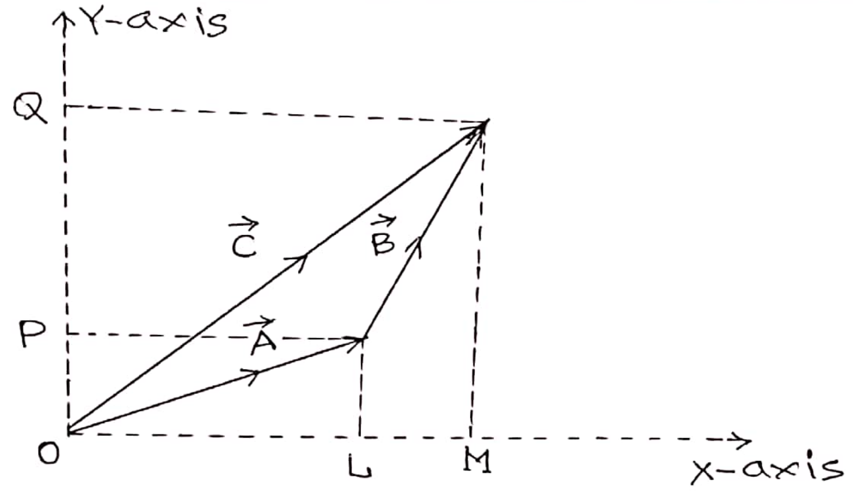
$\theta_1 = 30°$

Angle of $\vec{B}$ with x-axis

$\theta_2 = -40° = 320°$



Addition of vectors by resolution



In figure

$OL = A_x$, $LM = B_x$, $OM = C_x$

$OP = A_y$, $PQ = B_y$, $OQ = C_y$

A/c to triangle law

$$\vec{C} = \vec{A} + \vec{B}$$

As $OM = OL + LM$

$\therefore$ $C_x = A_x + B_x$

Also $OQ = OP + PQ$

$\therefore$ $C_y = A_y + B_y$

Clearly, when

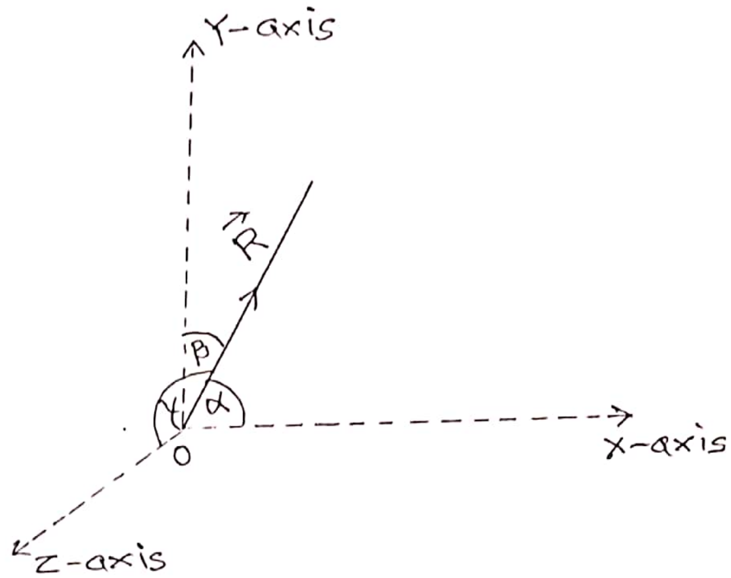
$\vec{R} = \vec{A} + \vec{B} - \vec{C}$, then

$$R_x = A_x + B_x - C_x$$

and

$$R_y = A_y + B_y - C_y$$

Resolution in space



Space में एक vector $\vec{R}$ को माना जो x-axis, Y-axis एवं Z-axis के साथ क्रमशः angles α, β एवं γ बनाते हैं।

$\vec{R}$ का x-component
$$R_x = |\vec{R}|\cos\alpha$$
$\vec{R}$ का y-component
$$R_y = |\vec{R}|\cos\beta$$
$\vec{R}$ का z-component
$$R_z = |\vec{R}|\cos\gamma$$

Component form of $\vec{R}$
$$\vec{R} = R_x\hat{i} + R_y\hat{j} + R_z\hat{k}$$

In magnitude
$$|\vec{R}| = \sqrt{R_x^2 + R_y^2 + R_z^2}$$

Direction of $\vec{R}$ with x-axis
$$\cos\alpha = \frac{R_x}{|\vec{R}|}, \quad \tan\alpha = \frac{\sqrt{R_y^2 + R_z^2}}{R_x}$$

Direction of $\vec{R}$ with y-axis
$$\cos\beta = \frac{R_y}{|\vec{R}|}, \quad \tan\beta = \frac{\sqrt{R_x^2 + R_z^2}}{R_y}$$

Direction of $\vec{R}$ with z-axis
$$\cos\gamma = \frac{R_z}{|\vec{R}|}, \quad \tan\gamma = \frac{\sqrt{R_x^2 + R_y^2}}{R_z}$$

Direction cosines (कोज्या दिशा)

Conventionally cos α, cos β and cos γ are denoted by l, m and n respectively and are called direction cosines. so

$$l = \cos\alpha, \quad m = \cos\beta, \quad n = \cos\gamma$$

मान्यता के अनुसार cos α, cos β एवं cos γ को क्रमशः l, m एवं n से दिखलाया जाता है तथा इन्हे कोज्या दिशा कहा जाता है।

Now
$$\vec{R} = R_x\hat{i} + R_y\hat{j} + R_z\hat{k}$$
$$= |\vec{R}|\cos\alpha\,\hat{i} + |\vec{R}|\cos\beta\,\hat{j} + |\vec{R}|\cos\gamma\,\hat{k}$$
$$= |\vec{R}|(\cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k})$$

$\Rightarrow \vec{R} = |\vec{R}|(l\hat{i} + m\hat{j} + n\hat{k})$

$\vec{R}$ की direction में एक unit vector

$$\hat{u} = \frac{\vec{R}}{|\vec{R}|} = l\hat{i} + m\hat{j} + n\hat{k}$$

Ex- जब $\vec{A} = 6m\hat{i} + 2m\hat{j} - 3m\hat{k}$ हो तब

(1) $\vec{A}$ का magnitude होगा

(A) 4m (B) 5m

(C) 6m (D) 7m Ans- (D)

(2) $\vec{A}$ का direction cosines होगा

(A) $l = \frac{2}{7}, m = \frac{6}{7}, n = -\frac{3}{7}$

(B) $l = \frac{6}{7}, m = \frac{2}{7}, n = -\frac{3}{7}$

(C) $l = \frac{3}{7}, m = \frac{2}{7}, n = \frac{6}{7}$

(D) None of these Ans- (B)

(3) $\vec{A}$ की direction में unit vector होगा

(A) $\frac{6}{7}\hat{i} + \frac{2}{7}\hat{j} + \frac{3}{7}\hat{k}$ (B) $\frac{6}{7}\hat{i} + \frac{2}{7}\hat{j} - \frac{3}{7}\hat{k}$

(C) $\frac{2}{7}\hat{i} + \frac{6}{7}\hat{j} - \frac{3}{7}\hat{k}$ (D) None of these Ans- (B)

Solⁿ $\vec{A} = 6m\hat{i} + 2m\hat{j} - 3m\hat{k}$

As $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

compare करने पर

$A_x = 6m, A_y = 2m, A_z = -3m$

$|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2} = \sqrt{6^2 + 2^2 + (-3)^2} = \sqrt{36+4+9}$

$= \sqrt{49}\,m = 7m$

Direction cosines

$l = \cos\alpha = \frac{A_x}{|\vec{A}|} = \frac{6}{7}$

$m = \cos\beta = \frac{A_y}{|\vec{A}|} = \frac{2}{7}$

$n = \cos\gamma = \frac{A_z}{|\vec{A}|} = \frac{-3}{7}$

Unit vector along $\vec{A}$

$\hat{u} = l\hat{i} + m\hat{j} + n\hat{k}$

$= \frac{6}{7}\hat{i} + \frac{2}{7}\hat{j} + \frac{-3}{7}\hat{k}$

$= \frac{6}{7}\hat{i} + \frac{2}{7}\hat{j} - \frac{3}{7}\hat{k}$

Multiplication of vector -

(A) With numerical value – When a vector is multiplied with a numerical value, then

(I) The resultant is also a vector.
(II) Magnitude changes
(III) Unit doesn't change
(IV) With positive numerical value direction doesn't change where with negative numerical value direction becomes opposite.

Ex – Let $\vec{a} = 2.4\,m$ along east

$5\vec{a} = 5(2.4\,m$ along east$)$
$= 12\,m$ along east

$-5\vec{a} = -5(2.4\,m$ along east$)$
$= -12\,m$ along east
$= 12\,m$ along west.

(B) With scalar – When a vector is multiplied with a scalar, then

(i) The resultant is also a vector.
(ii) Magnitude changes
(iii) Unit changes
(iv) Direction doesn't change

Ex – force $\vec{F} = m\vec{a}$
momentum $\vec{p} = m\vec{v}$
Impulse $\vec{I} = \vec{F}\Delta t$

Let mass $m = 5\,kg$
acceleration $\vec{a} = 2.4\,m\,s^{-2}$ along east
Force $\vec{F} = m\vec{a}$
$= (5\,kg)(2.4\,m\,s^{-2}$ along east$)$
$= 12\,N$ along east.

(C) Multiplication of vector with vector – a vector is multiplied with a vector in two process.

(i) scalar product

(ii) vector product

Scalar product/Dot product – It is process of multiplication of a vector with a vector in which the resultant is a scalar.

⇒ scalar product of $\vec{A}$ and $\vec{B}$ is written as $\vec{A}\cdot\vec{B}$ and is called vector $\vec{A}$ dot vector $\vec{B}$.

⇒ Let angle between $\vec{A}$ and $\vec{B}$ is θ then

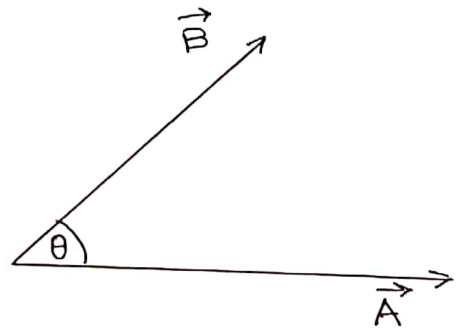
$$\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta$$

⇒ Scalar product of two vectors is equal to the product of magnitude of any one vector and the component of another vector along first vector.

⇒ In figure

$|\vec{B}|\cos\theta$ = component of $\vec{B}$ along $\vec{A}$

$|\vec{A}|\cos\theta$ = component of $\vec{A}$ along $\vec{B}$.



Where

$$\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A} = |\vec{A}||\vec{B}|\cos\theta$$

$$= |\vec{A}|(|\vec{B}|\cos\theta)$$

or

$$= |\vec{B}|(|\vec{A}|\cos\theta)$$

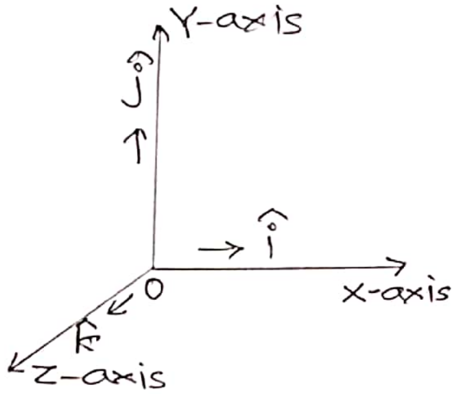
Scalar product of unit vectors

$\hat{i}\cdot\hat{i} = |\hat{i}||\hat{i}|\cos 0^\circ = (1)(1)(1) = 1$

$\therefore\ \hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

$\hat{i}\cdot\hat{j} = |\hat{i}||\hat{j}|\cos 90^\circ = (1)(1)(0) = 0$

$\therefore\ \hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$



Scalar product in component form

Let $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$

$\vec{A}\cdot\vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\cdot(B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$

$= (A_x B_x)\,\hat{i}\cdot\hat{i} + (A_x B_y)\,\hat{i}\cdot\hat{j} + (A_x B_z)\,\hat{i}\cdot\hat{k} + (A_y B_x)\,\hat{j}\cdot\hat{i}$
$+ (A_y B_y)\,\hat{j}\cdot\hat{j} + (A_y B_z)\,\hat{j}\cdot\hat{k} + (A_z B_x)\,\hat{k}\cdot\hat{i} + (A_z B_y)\,\hat{k}\cdot\hat{j}$
$+ (A_z B_z)\,\hat{k}\cdot\hat{k}$

$\therefore\ \vec{A}\cdot\vec{B} = A_x B_x + A_y B_y + A_z B_z$

Properties of scalar product –

(1) Scalar product of a vector by itself is equal to its square.
$\vec{A}\cdot\vec{A} = \vec{A}^2 = |\vec{A}|^2$

(2) Scalar product of two perpendicular vectors is 0
When $\vec{A}\perp\vec{B}$ then $\vec{A}\cdot\vec{B} = 0$

(3) When scalar product of two vectors is 0 then they are perpendicular
When $\vec{A}\cdot\vec{B} = 0$ then $\vec{A}\perp\vec{B}$

(4) Scalar product of identical unit vectors is equal to 1
$\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

(5) Scalar product of perpendicular unit vectors is 0

$$\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$$

(6) Scalar product follows commutative law

$$\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$$

(7) Scalar product follows associative law

$$\vec{A}\cdot(\vec{B}\cdot\vec{C}) = (\vec{A}\cdot\vec{B})\cdot\vec{C}$$

(8) Scalar product follows distributive law

$$\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$$

Uses of scalar product

(1) To find angle between two vectors

$\Rightarrow$ As $\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta$

So $\cos\theta = \dfrac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|}$

(2) To prove two vectors are perpendicular

$\Rightarrow$ As $\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta$

So when $\vec{A}\cdot\vec{B} = 0$ then $\vec{A}\perp\vec{B}$

(3) To find component of a vector along another vector.

$\Rightarrow$ As $\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta$, so

Component of $\vec{A}$ along $\vec{B} = \dfrac{\vec{A}\cdot\vec{B}}{|\vec{B}|} = \vec{A}\cdot\hat{B}$

component of $\vec{B}$ along $\vec{A} = \dfrac{\vec{A}\cdot\vec{B}}{|\vec{A}|} = \vec{B}\cdot\hat{A}$

Ex(1) – To find angle between $\vec{A} = 2\hat{i} + 3\hat{j} + 4\hat{k}$ and $\vec{B} = 4\hat{i} + 6\hat{j} + 8\hat{k}$.

(A) $0°$ (B) $180°$ (C) $90°$ (D) $45°$

Ans – (A) Soln. $\vec{A} = 2\hat{i} + 3\hat{j} + 4\hat{k} \Rightarrow |\vec{A}| = \sqrt{2^2+3^2+4^2} = \sqrt{29}$

$\vec{B} = 4\hat{i} + 6\hat{j} + 8\hat{k} \Rightarrow |\vec{B}| = \sqrt{4^2+6^2+8^2} = \sqrt{116}$

$\vec{A}\cdot\vec{B} = (2)(4) + (3)(6) + (4)(8)$

$= 8 + 18 + 32 = 58$

$$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}|\,|\vec{B}|} = \frac{58}{\sqrt{116\times 29}} = \frac{58}{\sqrt{58\times 2\times 29}} = \frac{58}{58} = 1$$

$\therefore \theta = 0°$

Ex(2) – To find angle between vectors $(\hat{i}+\hat{j})$ and $(\hat{i}+\hat{k})$

(A) $90°$ (B) $180°$ (C) $0°$ (D) $60°$

Ans – (D) Soln. $\vec{A} = \hat{i}+\hat{j} \Rightarrow |\vec{A}| = \sqrt{1^2+1^2} = \sqrt{2}$

$\vec{B} = \hat{i}+\hat{k} \Rightarrow |\vec{B}| = \sqrt{1^2+0^2+1^2} = \sqrt{2}$

$\vec{A}\cdot\vec{B} = (1)(1) + (1)(0) + (0)(1) = 1$

$$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}|\,|\vec{B}|} = \frac{1}{\sqrt{2}\sqrt{2}} = \frac{1}{2}$$

$\theta = 60°$

Ex(3) – When a force of 20N acts on a body a displacement of 6m comes at an angle of 60° with the direction of force. Work done by the force is

(A) 15 J (B) 30J (C) 60J (D) 120J

Ans – (C), Soln. Work $W = FS\cos 60° = (20)(6)(\frac{1}{2}) = 60J$

Ex(4) – Under force $\vec{F} = (6\hat{i} + 5\hat{j} - 3\hat{k})$ N displacement of a body is $\vec{S} = (2\hat{i} + 4\hat{j} + 4\hat{k})$ m. To find work done by the force

(A) 10J (B) 20J (C) 32 J (D) 44J

Ans – (B)

Soln. $W = \vec{F}\cdot\vec{S} = F_x S_x + F_y S_y + F_z S_z$

$= (6)(2) + (5)(4) + (-3)(4) = 20J$

Vector product

A process of multiplication of a vector with a vector in which the resultant is also a vector is known as vector product.

⇒ Vector product of vector $\vec{A}$ and $\vec{B}$ is ~~denot~~ written as $\vec{A} \times \vec{B}$ and spoken as vector $\vec{A}$ cross vector $\vec{B}$.

⇒ Let $\vec{C} = \vec{A} \times \vec{B}$

angle between $\vec{A}$ and $\vec{B}$ is θ, then

In magnitude

$$|\vec{C}| = |\vec{A}||\vec{B}| \sin\theta$$

With direction

$$\vec{C} = |\vec{A}||\vec{B}| \sin\theta \, \hat{n}$$

where $\hat{n}$ = a unit vector along $\vec{C}$.

Direction of vector product – direction of vector product is defined by right hand palm rule

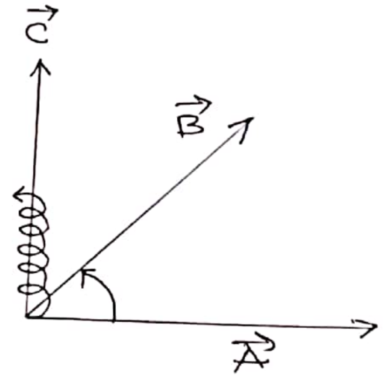
⇒ Resultant of vector product is perpendicular to both vectors.

⇒ According to right hand palm rule when stretched right hand palm is set perpendicular to the plane of two vector in such a way that initially fingers go towards first vector and after curling they go towards second vector then the direction of thumb gives the direction of vector product.

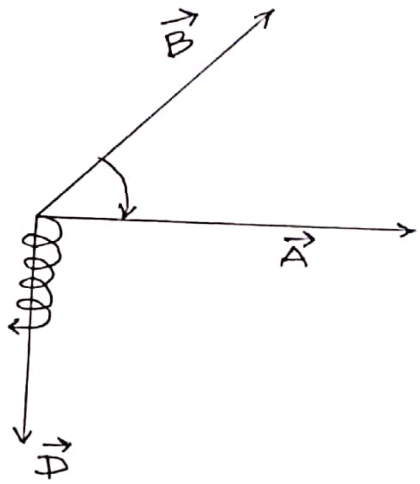
⇒ When curling of fingers from 1st vector to 2nd vector is anticlockwise then vector product is perpendicular outside the plane.

⇒ When curling of fingers from 1st vector to 2nd vector is clockwise, then resultant vector produced is perpendicular inside the plane of two vectors.

Let $\vec{C} = \vec{A} \times \vec{B}$



$\vec{D} = \vec{B} \times \vec{A}$



Examples of vector product

(1) Torque

$\vec{\tau} = \vec{r} \times \vec{F}$, where, $\vec{r}$ = displacement
$\vec{F}$ = force

(2) Angular momentum

$\vec{L} = \vec{r} \times \vec{P}$, where, $\vec{r}$ = displacement
$\vec{P}$ = Linear momentum

(3) Linear velocity

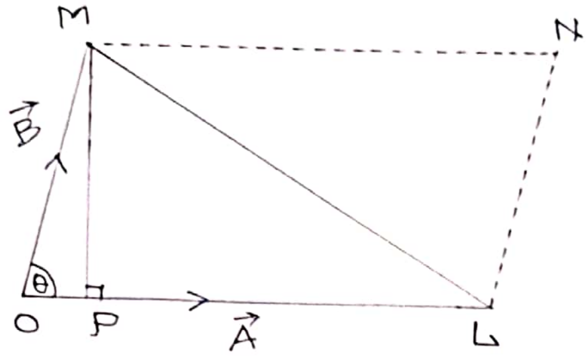
$\vec{V} = \vec{\omega} \times \vec{r}$, where, $\vec{\omega}$ = angular velocity
$\vec{r}$ = radial vector

(4) Linear acceleration

$\vec{a} = \vec{\alpha} \times \vec{r}$, where, $\vec{\alpha}$ = angular acceleration
$\vec{r}$ = radial vector

Geometrical interpretion of Vector product

In figure
$\vec{OL} = \vec{A}$
$\vec{OM} = \vec{B}$
θ = angle between $\vec{A}$ and $\vec{B}$
$MP \perp OL$



In right Δ OPM

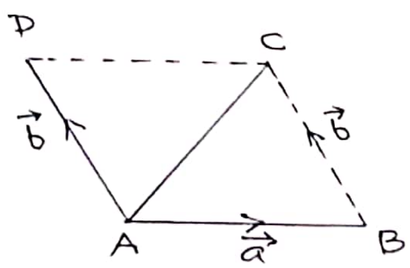
$$\sin\theta = \frac{PM}{OM} = \frac{PM}{|\vec{B}|}$$

$$\Rightarrow \quad PM = |\vec{B}|\sin\theta$$

Now $|\vec{A}\times\vec{B}| = |\vec{A}|\ |\vec{B}|\sin\theta$
$= (OL)(PM)$
$= 2\times\left[\frac{1}{2}\times(OL)\times(PM)\right]$
$= 2\times$ area of Δ OLM
$=$ area of parallelogram OLNM

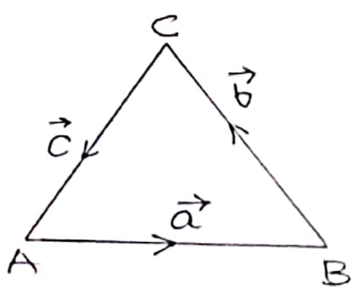
Clearly, magnitude of vector product is equal to the area of a parallelogram formed by the vectors as adjacent sides.

$\Rightarrow$



area of parallelogram ABCD $= |\vec{a}\times\vec{b}|$

$\therefore$ area of Δ ABC
$= \frac{1}{2}|\vec{a}\times\vec{b}|$



Area of Δ ABC
$= \frac{1}{2}|\vec{a}\times\vec{b}|$
$= \frac{1}{2}|\vec{b}\times\vec{c}|$
$= \frac{1}{2}|\vec{c}\times\vec{a}|$
$= \frac{1}{6}\left[|\vec{a}\times\vec{b}|+|\vec{b}\times\vec{c}|+|\vec{c}\times\vec{a}|\right]$

Vector product of unit vectors

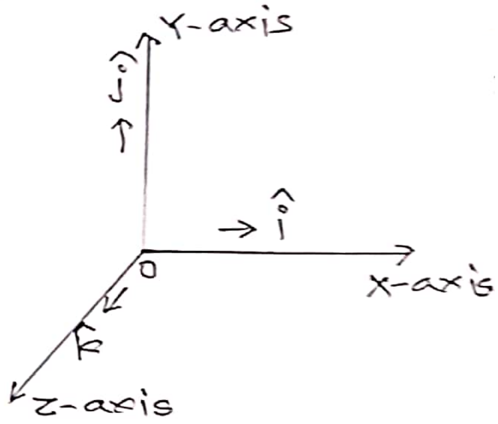
$\hat{i}\times\hat{i} = |\hat{i}||\hat{i}|\sin 0° = 0$

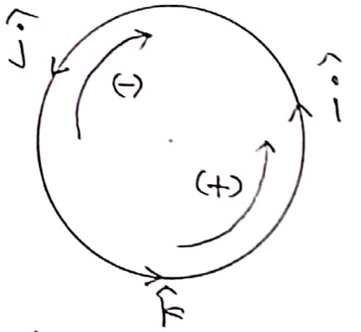
$\therefore\ \hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0$

$\hat{i}\times\hat{j} = |\hat{i}||\hat{j}|\sin 90°\ \hat{k}$

So

$\hat{i}\times\hat{j} = \hat{k}$	$\hat{j}\times\hat{i} = -\hat{k}$
$\hat{j}\times\hat{k} = \hat{i}$	$\hat{k}\times\hat{j} = -\hat{i}$
$\hat{k}\times\hat{i} = \hat{j}$	$\hat{i}\times\hat{k} = -\hat{j}$





Vector product in component form

Vector product in component form is solved by determinant as

Let $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$, then

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= \hat{i}\begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} - \hat{j}\begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} + \hat{k}\begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}$$

$$= (A_y B_z - B_y A_z)\hat{i} - (A_x B_z - B_x A_z)\hat{j} + (A_x B_y - B_x A_y)\hat{k}$$

Properties of vector product

(1) Vector product of a vector by itself is equal to zero vector

$$\vec{A} \times \vec{A} = \vec{0}$$

(2) Vector product of two parallel vectors is a zero vector

When $\vec{A} \parallel \vec{B}$, then $\vec{A} \times \vec{B} = \vec{0}$

(3) When vector product of two nonzero vectors is a zero vector then they are parallel.

When $\vec{A} \times \vec{B} = \vec{0}$ then $\vec{A} \parallel \vec{B}$

(4) Vector product of identical unit vectors is a zero vector

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$$

(5) Vector product of perpendicular unit vectors is also a unit vector.

$\hat{i} \times \hat{j} = \hat{k}$	$\hat{j} \times \hat{i} = -\hat{k}$
$\hat{j} \times \hat{k} = \hat{i}$	$\hat{k} \times \hat{j} = -\hat{i}$
$\hat{k} \times \hat{i} = \hat{j}$	$\hat{i} \times \hat{k} = -\hat{j}$

(6) Vector product doesn't follow commoulative law

$$\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$$

but $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$

(7) Vector product doesn't follow associative law

$$\vec{A} \times (\vec{B} \times \vec{C}) \neq (\vec{A} \times \vec{B}) \times \vec{C}$$

(8) Vector product follows distributive law

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

Uses of vector product

(1) To find angle between two vectors

As $|\vec{A}\times\vec{B}| = |\vec{A}||\vec{B}|\sin\theta$

$\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta$

So $\tan\theta = \dfrac{|\vec{A}\times\vec{B}|}{\vec{A}\cdot\vec{B}}$

(2) To prove two vectors are parallel

As $|\vec{A}\times\vec{B}| = |\vec{A}||\vec{B}|\sin\theta$

So when $\vec{A}\times\vec{B} = 0$, then

$|\vec{A}||\vec{B}|\sin\theta = 0$

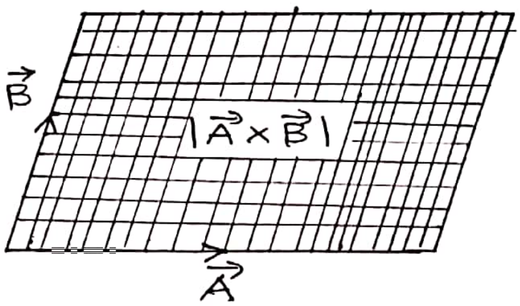
$\sin\theta = 0$

$\theta = 0^\circ$

$\therefore \vec{A} \parallel \vec{B}$

(3) To find area bounded by a geometrical construction made up of vectors.

$\Rightarrow |\vec{A}\times\vec{B}|$ = Area bounded by a parallelogram made up of vectors $\vec{A}$ and $\vec{B}$ as adjacent sides.



(4) To find component of a vector perpendicular to another vector.

$\Rightarrow$ Component of $\vec{A}$ perpendicular to $\vec{B}$

$$\vec{C} = \vec{A} - \frac{\vec{A}\cdot\vec{B}}{|\vec{B}|^2}\times\vec{B}$$

$$= \vec{A} - (\vec{A}\cdot\hat{B})\times\hat{B}$$

Triple scalar product (Triple dot product)

Triple scalar product of vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ is written as $\vec{A}\cdot(\vec{B}\times\vec{C})$ or $[\vec{A}\ \vec{B}\ \vec{C}]$

⇒ Resultant of triple scalar product is a scalar.

⇒ Triple scalar product in same order are equal

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$$

⇒ Triple scalar product in opposite order are opposite

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = -\vec{A}\cdot(\vec{C}\times\vec{B})$$

⇒ Let $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$

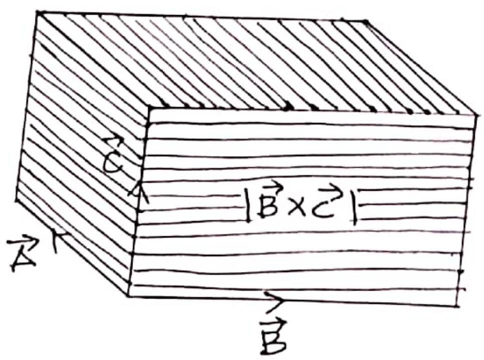
$\vec{C} = C_x\hat{i} + C_y\hat{j} + C_z\hat{k}$

then

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

⇒ When $\vec{A}\cdot(\vec{B}\times\vec{C}) = 0$ then $\vec{A}$, $\vec{B}$ and $\vec{C}$ are coplanar.

⇒ Triple scalar product is equal to volume of a parallelopiped made up of three vectors as adjacent sides.



$\vec{A}\cdot(\vec{B}\times\vec{C}) = |\vec{A}| \times$ component of $(\vec{B}\times\vec{C})$ along $\vec{A}$

= length × area

= Volume.

Triple vector product

Triple vector product of $\vec{A}$, $\vec{B}$ and $\vec{C}$ is written as $\vec{A} \times (\vec{B} \times \vec{C})$

⇒ Resultant of triple vector product is also a vector.

⇒ $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$

Similarly

$\vec{B} \times (\vec{C} \times \vec{A}) = (\vec{B} \cdot \vec{A})\vec{C} - (\vec{B} \cdot \vec{C})\vec{A}$

$\vec{C} \times (\vec{A} \times \vec{B}) = (\vec{C} \cdot \vec{B})\vec{A} - (\vec{C} \cdot \vec{A})\vec{B}$

Ex – When $\vec{A}$, $\vec{B}$ and $\vec{C}$ are mutual perpendicular then to prove $\vec{A} \times (\vec{B} \times \vec{C}) = 0$.

Solⁿ. As $\vec{A} \perp \vec{B} \perp \vec{C}$

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$$
$$= (|\vec{A}||\vec{C}|\cos 90^\circ)\vec{B} - (|\vec{A}||\vec{B}|\cos 90^\circ)\vec{C}$$
$$= 0 - 0$$
$$= 0$$

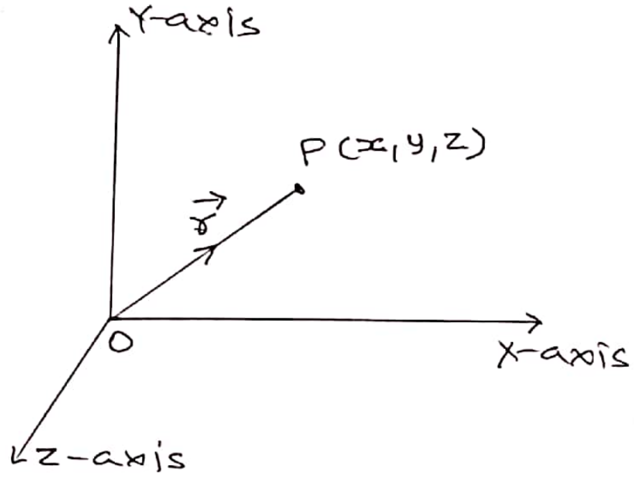
Position vector – a line-segment from origin/reference point to the position of a point is known as position vector.

Suppose position of a particle is $P(x, y, z)$ then position vector

$$\vec{r} = \vec{OP} = x\hat{i} + y\hat{j} + z\hat{k}$$

In magnitude

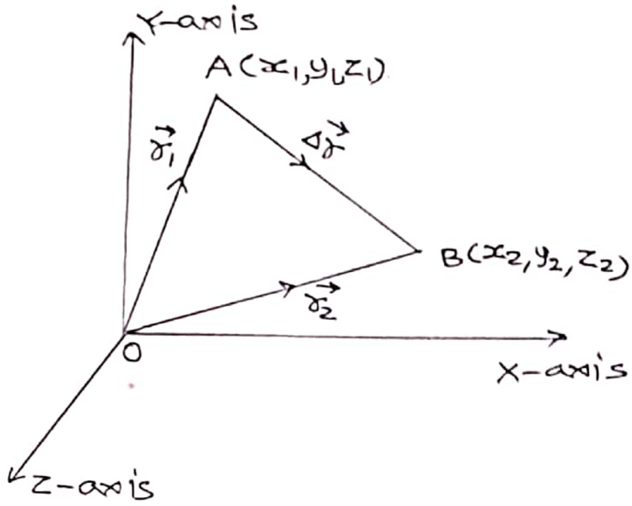
$$|\vec{r}| = r = \sqrt{x^2 + y^2 + z^2}$$



Displacement

A line segment from initial position to final position is known as displacement.

Suppose position of a particle changes from point $A(x_1, y_1, z_1)$ to point $B(x_2, y_2, z_2)$ then



Initial position vector

$$\vec{r_1} = \vec{OA} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$$

Final position vector

$$\vec{r_2} = \vec{OB} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$$

Displacement

$$\Delta\vec{r} = \vec{AB}$$

According to triangle law

$$\vec{OA} + \vec{AB} = \vec{OB}$$

$$\Rightarrow \quad \vec{AB} = \vec{OB} - \vec{OA}$$

$\therefore \quad \Delta\vec{r} = \vec{r_2} - \vec{r_1}$ = final position vector − initial position vector.

So $\Delta\vec{r} = (x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k})$

$$\therefore \quad \Delta\vec{r} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

In magnitude

$$|\Delta\vec{r}| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

Ex- Position of a particle changes from (2m, −1m, 3m) to (5m, 4m, 6m). Displacement of the particle is

(A) $(3\hat{i} + 5\hat{j} + 9\hat{k})$ m (B) $(6\hat{i} + 5\hat{j} + 3\hat{k})$ m

(C) $(3\hat{i} + 5\hat{j} + 3\hat{k})$ m (D) None of these

Ans- (C)

Sol^n $\Delta\vec{r} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$

$= (5-2)\hat{i} + (4+1)\hat{j} + (6-3)\hat{k}$

$= (3\hat{i} + 5\hat{j} + 3\hat{k})$ m